

REPORT DOCUMENTATION PAGE			Form Approved OMB No. 0704-0188	
Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.				
1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE October 1992	3. REPORT TYPE AND DATES COVERED Reprint		
4. TITLE AND SUBTITLE  Exploitation of Higher-Order Cyclostationarity for Weak-Signal Detection and Time-Delay Estimation		5. FUNDING NUMBERS  DAAL03-91-C-0018		
6. AUTHOR(S)  Chad M. Spooner and William A. Gardner				
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)  Statistical Signal Processing, Inc. 6950 Yount Street Yountville, CA 94599		8. PERFORMING ORGANIZATION REPORT NUMBER		
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)  U.S. Army Research Office P. O. Box 12211 Research Triangle Park, NC 27709-2211		10. SPONSORING/MONITORING AGENCY REPORT NUMBER  ARO 27917.6-EK-5		
11. SUPPLEMENTARY NOTES The view, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy, or decision, unless so designated by other documentation.				
12a. DISTRIBUTION/AVAILABILITY STATEMENT  Approved for public release; distribution unlimited.			12b. DISTRIBUTION CODE	
13. ABSTRACT (Maximum 200 words)  The cumulant theory of cyclostationary time-series is applied to several types of weak-signal detection problems that arise in the area of signal interception, and to the problem of estimating the relative time-delay of a heavily corrupted signal that is received at two locations. The cumulant theory of cyclostationarity (CS) is the theory of higher-order temporal and spectral cumulants and moments of CS time-series. Specifically, the theory characterizes the additive sine waves present in the output of nonlinear transformations of CS time-series. The detection and time-delay estimation problems that are posed are difficult to solve because the signal is weak, the noise and interference is nonstationary and non-Gaussian, and the signal does not exhibit second-order CS.				
14. SUBJECT TERMS  Time-delay estimation, signal detection, cyclostationarity, higher-order statistics			15. NUMBER OF PAGES 5	
			16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL	

# Exploitation of Higher-Order Cyclostationarity for Weak-Signal Detection and Time-Delay Estimation \*

Chad M. Spooner William A. Gardner  
 Department of Electrical and Computer Engineering  
 University of California  
 Davis, CA 95616

By _____	
Distribution / _____	
Availability Codes	
Dist	Avail and/or Special
A-1	

## Abstract

*The cumulant theory of cyclostationary time-series is applied to several types of weak-signal detection problems that arise in the area of signal interception, and to the problem of estimating the relative time-delay of a heavily corrupted signal that is received at two locations. The cumulant theory of cyclostationarity (CS) is the theory of higher-order temporal and spectral cumulants and moments of CS time-series. Specifically, the theory characterizes the additive sine waves present in the output of nonlinear transformations of CS time-series. The detection and time-delay estimation problems that are posed are difficult to solve because the signal is weak, the noise and interference is nonstationary and non-Gaussian, and the signal does not exhibit second-order CS.*

## 1 INTRODUCTION

The problems considered in this paper can be formulated in terms of the following two-sensor signal model

$$x(t) = s(t) + m_x(t) + \sum_{k=1}^M i_k(t) \quad (1)$$

$$y(t) = A_0 s(t + d_0) + m_y(t) + \sum_{k=1}^M A_k i_k(t + d_k),$$

where  $s(t)$  is the signal of interest (SOI),  $m_x(t)$  and  $m_y(t)$  are independent white Gaussian noises, the  $\{i_k(t)\}$  are signals not of interest (SNOIs),  $\{A_k\}$  are attenuation factors, and  $\{d_k\}$  are the relative delays between the signal components in  $x(t)$  and  $y(t)$ . For example, suppose that  $M = 2$  and the power levels of  $m_x$  and  $m_y$  are time-varying. Further, assume that the SOI is weak and fourth-order cyclostationary (CS)

– but not second-order CS [8] – with period  $T_0$ , and the two interferers are second-order CS with periods  $T_1$  and  $T_2$  such that  $T_1 + T_2 = T_0$ . The first problem is to detect the presence of the SOI given a finite segment of  $x(t)$ . The next problem is to determine the parameter  $d_0$  given finite segments of both  $x(t)$  and  $y(t)$ . These problems are difficult to solve using the stationary models of the various signals (which leads to radiometry [4] and generalized cross correlation methods [6]) because for detection the nonstationary noise and interference complicates the threshold setting, and for time-delay estimation the interference corrupts the relevant phase information in the cross spectrum. The theory of higher-order statistics is not helpful because it also is not signal selective: all of the signals contribute to higher-order cumulants for a stationary signal model [8]. The theory of second-order CS (SOCS) is also not helpful because the SOI has no second-order cyclic features [3]. However, these two tasks (and other similar tasks) can be handled by using the higher-order CS (HOCS) of the SOI. Before showing how this can be done, we first review the definitions of higher-order moments and cumulants for CS time-series.

## 2 THE PARAMETERS OF HOCS

For the time-series  $x(t)$  for  $-\infty < t < \infty$ , we define the  $n$ th-order lag-product time-series by

$$L_x(t, \tau)_n \triangleq \prod_{j=1}^n x(t + \tau_j), \quad \tau \triangleq [\tau_1 \cdots \tau_n]^\dagger, \quad (2)$$

where  $\dagger$  denotes transposition. The  $n$ th-order *cyclic temporal moment function* (CTMF) is defined by the limiting time average

$$\begin{aligned} R_x^\alpha(\tau)_n &\triangleq \lim_{T \rightarrow \infty} \frac{1}{T} \int_{v-T/2}^{v+T/2} L_x(t, \tau)_n e^{-i2\pi\alpha t} dt \\ &\equiv \langle L_x(t, \tau)_n e^{-i2\pi\alpha t} \rangle_0, \end{aligned} \quad (3)$$

\*This work was jointly supported by the National Science Foundation under grant MIP-91-12800 and the Army Research Office under contract DAAL03-91-C-0018.

and is simply the Fourier coefficient associated with the component  $e^{i2\pi\alpha t}$  in the time-series  $L_x(t, \tau)_n$ . It can be seen that the CTMF is a Fourier coefficient of a moment function because the  $n$ th-order fraction-of-time probabilistic moment (the *temporal moment function*) associated with  $L_x(t, \tau)_n$  can be expressed as [1]

$$R_x(t, \tau)_n = \sum_{\alpha} R_x^{\alpha}(\tau)_n e^{i2\pi\alpha t}, \quad (4)$$

where the sum is over all real numbers  $\alpha$ , called  $n$ th-order *impure cycle frequencies*, for which  $R_x^{\alpha}(\tau)_n \neq 0$ . The functions (3) and (4) exist and are well-behaved for appropriate models of many time-series including amplitude modulated, pulse-amplitude-modulated, and phase-shift-keyed and frequency-shift-keyed signals [8].

The  $n$ th-order *temporal cumulant function* (TCF) for the set of time-series translates  $\{x(t + \tau_j)\}_{j=1}^n$  is defined by (cf. [7] and [8])

$$C_x(t, \tau)_n = \sum_{P=\{\nu_k\}_{k=1}^p} \left[ k(p) \prod_{j=1}^p R_x(t, \tau_{\nu_j})_{n_j} \right]. \quad (5)$$

The sum in (5) is over the set  $P$  of all distinct partitions of the index set  $\{1, 2, \dots, n\}$ , where each partition  $\{\nu_k\}_{k=1}^p$  has  $p$  elements,  $1 \leq p \leq n$ , and  $k(p) = (-1)^{p-1}(p-1)!$ . The vector  $\tau_{\nu_j}$  consists of the  $n_j$  lags with indices in the set  $\nu_j$ . The *cyclic temporal cumulant function* (CTCF) is the Fourier coefficient of the TCF:

$$C_x^{\beta}(\tau)_n \triangleq \langle C_x(t, \tau)_n e^{-i2\pi\beta t} \rangle_0, \quad (6)$$

where  $\beta$  is an  $n$ th-order *pure cycle frequency* for  $x(t)$  [2, 8]. Combining (4)–(6) reveals that the CTCF is given by the following explicit function of lower-order CTMFs:

$$C_x^{\beta}(\tau)_n = \sum_{P=\{\nu_k\}_{k=1}^p} \left[ k(p) \sum_{\alpha^{\dagger} \mathbf{1} = \beta} \prod_{j=1}^p R_x^{\alpha_j}(\tau_{\nu_j})_{n_j} \right],$$

where  $\mathbf{1} = [1 \dots 1]^{\dagger}$ , and  $\alpha = [\alpha_1 \dots \alpha_p]^{\dagger}$ . An  $n$ th-order pure cycle frequency, as first defined in [2], is the frequency of a finite-strength additive sine-wave component in  $L_x(t, \tau)_n$  that is free of all contributions from products of sine-wave components from lower-order lag products obtained by factoring  $L_x(t, \tau)_n$ .

The CTMF and the CTCF are not in general absolutely integrable due to the presence of sinusoidal components in  $\tau$ . These components formally result in Dirac deltas in the  $n$ -dimensional Fourier transform of the CTCF. However, a reduced-dimension (RD) version of the CTCF is absolutely integrable

for many time-series of interest and, therefore, it is strictly Fourier transformable [8]. The RD-CTCF is simply the CTCF associated with the  $n$  variables  $\{x(t + \tau_j)\}_{j=1}^n$  with  $\tau_n = 0$ . We use the notation

$$\bar{C}_x^{\beta}(\mathbf{u})_n \triangleq C_x^{\beta}(\tau)_n \quad \tau = [\mathbf{u} \ 0], \quad (7)$$

where  $\mathbf{u}$  is an  $(n-1)$ -dimensional vector. The  $(n-1)$ -dimensional Fourier transform of (7) is denoted by  $\bar{P}_x^{\beta}(\mathbf{f}')_n$ :

$$\bar{P}_x^{\beta}(\mathbf{f}')_n = \int_{-\infty}^{\infty} \bar{C}_x^{\beta}(\mathbf{u})_n e^{-i2\pi\mathbf{u}^{\dagger} \mathbf{f}'} d\mathbf{u}, \quad (8)$$

where  $\mathbf{f}' = [f_1 \dots f_{n-1}]^{\dagger}$ .

Consider the  $n$  complex-demodulate time-series  $X_T(t, f_j)$  for  $j = 1, \dots, n$ , associated with narrow bandpass filtered versions of  $x(t)$ , where

$$X_T(t, f) = \int_{t-T/2}^{t+T/2} x(v) e^{-i2\pi f v} dv. \quad (9)$$

The limit as  $T \rightarrow \infty$  of the limiting time-average of the product of these spectral components is called the *spectral moment function*

$$S_x(\mathbf{f})_n \triangleq \lim_{T \rightarrow \infty} \left\langle \prod_{j=1}^n X_T(t, f_j) \right\rangle_0, \quad (10)$$

and it can be shown that Dirac deltas (denoted by  $\delta(\cdot)$ ) can be factored out as follows:

$$S_x(\mathbf{f})_n = \sum_{\alpha} \bar{S}_x^{\alpha}(\mathbf{f}')_n \delta(\mathbf{f}^{\dagger} \mathbf{1} - \alpha). \quad (11)$$

However, the factor  $\bar{S}_x^{\alpha}(\mathbf{f}')_n$  typically contains additional Dirac deltas [8]. The *spectral cumulant function* is given by

$$P_x(\mathbf{f})_n = \sum_{P=\{\nu_k\}_{k=1}^p} \left[ k(p) \prod_{j=1}^p S_x(\mathbf{f}_{\nu_j})_{n_j} \right], \quad (12)$$

where  $\mathbf{f}_{\nu_j}$  is the vector of frequencies with subscripts in the set  $\nu_j$ , and it follows from (8) and (11) that Dirac deltas can be factored out:

$$P_x(\mathbf{f})_n = \sum_{\beta} \bar{P}_x^{\beta}(\mathbf{f}')_n \delta(\mathbf{f}^{\dagger} \mathbf{1} - \beta), \quad (13)$$

where  $\bar{P}_x^{\beta}(\mathbf{f}')_n$  is called the *cyclic polyspectrum* (CP) and does not contain Dirac deltas. As first shown in [2], the CP is the  $(n-1)$ -dimensional Fourier transform of the RD-CTCF (8). This is a generalization of the Wiener relation between the power spectrum and autocorrelation from second-order stationary time-series

to  $n$ th-order CS time-series (cf. [1]). For strictly stationary time-series, the only cycle frequency (pure or impure) is  $\alpha = \beta = 0$ . For a strictly stationary stochastic process, the same is true for probabilistic versions of the temporal moments and cumulants defined in this section, which form the basis for the theory of higher-order statistics. Such parameters are not *signal selective*, as noted in the next section, whereas the cyclic parameters ( $\beta \neq 0, \alpha \neq 0$ ), which form the basis for the theory of HOCS, can be signal selective.

### 3 SIGNAL SELECTIVITY

An important advantage of exploiting CS in signal processing tasks is that the cyclic parameters are signal selective in that the parameters associated with the SOI can be estimated from data that also contains noise and SNOIs; and as the amount of data becomes infinite, the effects on the estimate of the noise and interference vanish. The nature of the signal selectivity properties of higher-order cyclic moments and cumulants is examined next.

Because the signals and noises in (1) are assumed to be mutually independent, the TCF for  $x(t)$  is given by the simple formula [8]

$$C_x(t, \tau)_n = C_s(t, \tau)_n + C_{m_x}(t, \tau)_n + \sum_{k=1}^M C_{i_k}(t, \tau)_n,$$

which implies that if  $\beta$  is an  $n$ th-order pure cycle frequency for  $s(t)$ , and is not for any of the other signals (it is unique to  $s(t)$ ), then  $C_x^\beta(\tau)_n = C_s^\beta(\tau)_n$ . Similarly, if  $\beta_k$  is a unique  $n$ th-order pure cycle frequency for  $i_k(t)$ , then  $C_{i_k}^{\beta_k}(\tau)_n = C_s^\beta(\tau)_n$ .

For moments, the signal selectivity property depends on the cycle frequencies for all of the signals present for all orders  $m \leq n$ . This can be seen by expressing the CTMF in terms of CTCFs [8]:

$$R_x^\alpha(\tau)_n = \sum_{P=\{\nu_k\}_{k=1}^n} \left[ \sum_{\beta^T \mathbf{1} = \alpha} \prod_{j=1}^n C_x^{\beta_j}(\tau_{\nu_j})_{n_j} \right]. \quad (14)$$

If there is a vector  $\beta$  such that at least one of its elements is a cycle frequency that is associated with a SNOI, then the CTMF for  $x(t)$  will not be equal to the CTMF for  $s(t)$ . Nevertheless, it can happen that the contributions to the CTMF from SNOIs do not affect the phase of the CTMF but only its magnitude. For moments, then, there are two kinds of signal selectivity, depending on what information in the moment is considered useful (magnitude or phase). Because of this potentially troublesome complication, the signal-selectivity properties of cumulants are deemed more

useful than those for moments. Thus, the following sections describe algorithms in terms of cumulants, but in most cases, an alternative algorithm can be created by simply replacing cumulants with moments. The usefulness of the resulting alternatives depends on the cycle frequencies associated with the SNOIs, which must be evaluated on a case-by-case basis.

### 4 WEAK-SIGNAL DETECTION

In this section, the problem of detecting the presence of the signal  $s(t)$  in a received data set  $x(t)$  as in (1) is considered. There are several versions of this detection problem that are of interest. The first is called the *general search problem*, in which a data set is analyzed to determine if there are *any* CS signals present. No information about the received data is assumed to be known in the general search problem. In the second problem, called the *known-cycle-frequency problem*, a specific pure cycle frequency/order pair  $(\beta, n_0)$  is of interest, and it is desired to determine if there is a signal present in the data corresponding to this pair. In the third problem, called the *known-modulation problem*, the modulation format of the signal of interest is known, and hence the cyclic cumulants of the signal are known (in principle); it is desired to determine the presence or absence of this signal.

#### The General Search Problem

In this problem there is a maximum order  $N$  of non-linearity that is to be used for processing. The goal of the processing is to produce a list of pure cycle frequencies  $\{\beta_n\}$  for each order  $n \leq N$ . This list  $\{\beta_n\}$  for each  $n$  characterizes the detectable CS of order  $n$  (and only  $n$ ) that is associated with  $x(t)$ . Thus, these lists are not contaminated by entries that are due to lower-order sine wave interactions. To accomplish this task, the TCF is estimated for  $x(t)$  for each order  $n$ . From the estimate of the TCF of order  $n$ , the cycle frequencies  $\{\beta_n\}$ , which are needed for the estimate of the TCF for order  $n+1$ , can be found. More explicitly, the general search problem can be tackled using the following algorithm:

0. Let  $n = 1$
1.  $\hat{C}'_x(t, \tau)_n = L_x(t, \tau)_n - \sum_{\substack{p \\ p \neq 1}} \hat{C}_x(t, \tau_{\nu_j})_{n_j}$
2.  $Y(f) = \text{FFT}_t \{ \hat{C}'_x(t, \tau)_n \}$
3. Threshold detect the bins of  $Y$  to find  $\{\beta_n\}$
4.  $\hat{C}_x^{\beta_n}(\tau)_n = \langle \hat{C}'_x(t, \tau)_n e^{-i2\pi\beta_n t} \rangle_0$
5.  $\hat{C}_x(t, \tau)_n = \sum_{\beta_n} \hat{C}_x^{\beta_n}(\tau)_n e^{i2\pi\beta_n t}$
6.  $n \rightarrow n+1$ ; if  $n \leq N$  then go to 1.

In step 4, the interval over which the average  $\langle \cdot \rangle_0$  is performed is determined by the amount of data available. If any of the detected cycle frequencies are of particular interest, a cyclic polyspectral analysis can be performed from which the modulation type can possibly be determined [8].

#### The Known-Cycle-Frequency Problem

In this problem, one or more of the signal's modulation frequencies, such as a symbol rate or carrier frequency, is assumed to be known, but the exact functional form of the CTCF is unknown. The environment is still assumed to be unknown and, therefore, the general search algorithm is still of interest. However, it can be improved for the known-cycle-frequency problem by combining it with a least-squares estimation technique. Let  $(\beta, n_0)$  be the cycle frequency/order pair of interest. Use the general search algorithm up to order  $n_0 - 1$ . Form  $\hat{C}'_x(t, \tau)_{n_0}$ , and use a least-squares estimator to detect the presence of the signal of interest using the statistic

$$Y = \langle \hat{w}^\dagger \hat{C}'_x(t, \tau)_{n_0} e^{-i2\pi\beta t} \rangle_0 = \hat{w}^\dagger \hat{C}_x^\beta(\tau)_{n_0},$$

where

$$\begin{aligned} \hat{C}'_x(t, \tau)_{n_0} &= [\hat{C}'_x(t, \tau_1)_{n_0} \cdots \hat{C}'_x(t, \tau_K)_{n_0}]^\dagger, \\ \hat{C}_x^\beta(\tau)_{n_0} &= [\hat{C}_x^\beta(\tau_1)_{n_0} \cdots \hat{C}_x^\beta(\tau_K)_{n_0}]^\dagger, \end{aligned}$$

and where  $\hat{w}$  is the least-squares weight vector

$$\hat{w} = \arg \min_{\hat{w}} \left\langle \left| \hat{w}^\dagger \hat{C}'_x(t, \tau)_{n_0} - e^{i2\pi\beta t} \right|^2 \right\rangle_0. \quad (15)$$

The solution to (15) is  $\hat{w} = R^{-1} \hat{C}_x^\beta(\tau)_{n_0}$ , where

$$R = \langle \hat{C}'_x(t, \tau)_{n_0} \hat{C}'_x(t, \tau)_{n_0}^H \rangle_0,$$

in which  $H$  denotes conjugate transpose. Thus, the detection statistic is

$$Y = \hat{C}_x^\beta(\tau)_{n_0}^H R^{-1} \hat{C}_x^\beta(\tau)_{n_0},$$

which is obtained by forming the particular linear combination of data sets  $\hat{C}'_x(t, \tau_1)_{n_0}, \dots, \hat{C}'_x(t, \tau_K)_{n_0}$  that optimally combines the regenerated sine waves with frequency  $\beta$  present in each set, and then correlates this composite regenerated sine wave with the stored sine wave  $e^{i2\pi\beta t}$ .

#### The Known-Modulation Problem

In this problem, it is desired to determine if a signal with known modulation type is present. In particular, the CTCF of  $s(t)$  for  $n = n_0$  and pure cycle frequency  $\beta$  is known. The general search algorithm can be used

to remove all lower-order sine waves up to order  $n_0 - 1$ . Then, from  $\hat{C}'_x(t, \tau)_{n_0}$  the CTCF estimate  $\bar{C}_{x_T}^\beta(u)_{n_0}$  for cycle frequency  $\beta$  can be determined by computing the Fourier coefficient as in (6). The proposed detection statistic is

$$Y = \int_{-\infty}^{\infty} \bar{C}_{x_T}^\beta(u)_{n_0} \bar{C}_s^\beta(u)_{n_0}^* du.$$

The primary justification for this particular statistic is that when no signal is present with  $n$ th-order pure cycle frequency  $\beta$ , then  $\bar{C}_{x_T}^\beta(u)_{n_0} \rightarrow 0$ , which implies that  $Y \rightarrow 0$ ; when the signal of interest is present, then

$$Y \rightarrow \int_{-\infty}^{\infty} |\bar{C}_s^\beta(u)_n|^2 du. \quad (16)$$

Thus,  $y$  is an asymptotically noise-free statistic on both the signal-present and signal-absent hypotheses. Furthermore, the integral (16) is finite [8]. Hence, this statistic is the natural generalization of the single cycle detector that exploits SOCS [4].

The detection statistic  $Y$  can be generalized to include only a portion of  $u$ -space, denoted by  $G \subset \mathbf{R}^{n_0}$ ,

$$Y = \int_G \bar{C}_{x_T}^\beta(u)_{n_0} \bar{C}_s^\beta(u)_{n_0}^* du.$$

Choices for  $G$  might include those values of  $u$  for which the RD-CTCF  $\bar{C}_s^\beta(u)_{n_0}$  is particularly large, or for which the coefficient of variation (variance divided by squared mean) of the estimator  $\bar{C}_{x_T}^\beta(u)_{n_0}$  of the RD-CTCF is particularly small [8].

## 5 TIME-DELAY ESTIMATION

Conventional approaches to the problem of estimating the time-delay (or time difference of arrival (TDOA)) between signal components in data from two sensors can be collectively referred to as *generalized cross correlation* (GCC) methods [6]. In the GCC methods, filtered versions of the sensor outputs  $x(t)$  and  $y(t)$  are cross correlated, and the estimate of  $d_0$  is taken to be the location of the peak in the cross-correlation estimate. These methods suffer when interferers are present ( $M \geq 1$  in (1)), because each interferer contributes a peak of its own to the cross correlation function. This causes two problems. The first is a resolution problem which, to be solved, requires that the differences in the TDOAs for each of the signals be greater than the widths of the cross correlation functions so that the peaks can be resolved. The second problem is that it is difficult to correctly associate each peak with its corresponding signal. Both of these problems arise because the GCC methods are not signal selective; they produce TDOA peaks for

all the signals in the received data unless they are spectrally disjoint and can, therefore, be separated by filtering. Signal-selective methods that exploit the SOCS of the desired signal, which is assumed to be unique to that signal, are studied in [5]. These methods have been shown to outperform the GCC methods, and have been shown to produce unbiased TDOA estimates with variance that is smaller than the Cramer-Rao lower bound on the variance of TDOA estimators that are based on the assumption that the signal and its environment are stationary. However, these methods fail when there is no SOCS to exploit. In this case, the theory of HOCS can be used to develop signal-selective TDOA estimators. Following the approach in [5] for SOCS, the methodology considered here for HOCS is based on least-squares estimation. The following two examples illustrate the methodology [8].

Define a cross cumulant between  $n-1$  time-translates of  $x(t)$  and one translate of  $y(t)$  as follows

$$C_{xy}(t, \tau)_n \triangleq \text{Cumulant} \{y(t + \tau_n), x(t + \tau_j)\}_{j=1}^n$$

The Fourier coefficient of this cross cumulant for the cycle frequency  $\beta$  for the signal model (1) (assuming that the noise and interference do not exhibit  $n$ th-order CS with pure cycle frequency  $\beta$ ) is given by

$$\begin{aligned} C_{xy}^\beta(\tau)_n &\triangleq \langle C_{xy}(t, \tau)_n e^{-i2\pi\beta t} \rangle_0 \\ &= A_0 C_s^\beta(\tau + \delta_n d_0)_n, \end{aligned}$$

where  $\delta_n$  is the unit vector along the  $n$ th coordinate. It is easy to show that the following relations involving RD-CTCFs hold:

$$\bar{C}_{xy}^\beta(u)_n = A_0 \bar{C}_s^\beta(u - \mathbf{1}d_0)_n e^{i2\pi\beta d_0}; \quad \bar{C}_x^\beta(u)_n = \bar{C}_s^\beta(u)_n.$$

This suggests a least-squares fit of a measurement of  $\bar{C}_{xy}^\beta$  to a measurement of  $\bar{C}_x^\beta$  over a region  $G$  of  $u$ -space of interest:

$$\min_{A, d} \int_G \left| \bar{C}_{xy}^\beta(u)_n - A \bar{C}_x^\beta(u - \mathbf{1}d)_n e^{i2\pi\beta d} \right|^2 du,$$

which leads to the following estimator of the delay  $d_0$ :

$$\hat{d}_0 = \arg \max_d \Re \int_G \bar{C}_{xT}^\beta(u)_n \bar{C}_{xyT}^\beta(u + \mathbf{1}d)_n^* e^{i2\pi\beta d} du.$$

This estimator is a higher-order generalization of the SPECTral COherence Alignment algorithm for TDOA estimation [5], which exploits SOCS, and has been shown to possess several optimality properties.

As an alternative, cross-sensor measurements can be avoided entirely by noting that

$$\bar{C}_y^\beta(u)_n = A_0^n \bar{C}_s^\beta(u)_n e^{i2\pi\beta d_0}; \quad \bar{C}_x^\beta(u)_n = \bar{C}_s^\beta(u)_n,$$

which suggests the following least-squares approach:

$$\hat{d}_0 = \arg \min_{A, d} \int_G \left| \bar{C}_{yT}^\beta(u)_n - A^n \bar{C}_{xT}^\beta(u)_n e^{i2\pi\beta d} \right|^2 du.$$

The estimator for  $d_0$  is given explicitly by

$$\hat{d}_0 = \frac{-1}{2\pi\beta} \int_G \bar{C}_{xT}^\beta(u)_n \bar{C}_{yT}^\beta(u)_n^* du, \quad (17)$$

which is a higher-order generalization of the second-order Cyclic Phase Difference algorithm for TDOA estimation without cross-sensor measurements [5].

## 6 CONCLUSIONS

The higher-order cumulants and moments of cyclostationary time-series can be used to perform difficult detection and estimation tasks. The basic signal-selectivity properties of cumulants yield estimators that are tolerant to noise and cyclostationary interference. This admittedly terse presentation will be expanded on in a forthcoming journal paper (see [8]).

## References

- [1] W. A. Gardner. *Statistical Spectral Analysis: A Non-probabilistic Theory*. Englewood Cliffs, New Jersey: Prentice-Hall 1987.
- [2] W. A. Gardner and C. M. Spooner. "Higher-Order Cyclostationarity, Cyclic Cumulants, and Cyclic Polyspectra," in *Proceedings of the International Symposium on Information Theory and its Applications (ISITA)*, Honolulu, Hawaii, November 1990.
- [3] W. A. Gardner. "Exploitation of Spectral Redundancy in Cyclostationary Signals," *IEEE Signal Processing Magazine*, Vol. 8, pp. 14-36, April, 1991.
- [4] W. A. Gardner. "Signal Interception: A Unifying Theoretical Framework for Feature Detection," *IEEE Trans. Comm.*, Vol. 36, pp. 897-906, August 1988.
- [5] W. A. Gardner and C. K. Chen. "Signal-Selective Time-Difference-of-Arrival Estimation for Passive Location of Man-made Signal Sources in Highly Corruptive Environments: Parts I and II," *IEEE Trans. Signal Proc.*, Vol. 40, pp. 1168-1197, 1992.
- [6] C. H. Knapp and G. C. Carter. "The Generalized Correlation Method for Estimation of Time Delay," *IEEE Transactions on Acoustics, Speech, and Signal Processing*, Vol. 24, August 1976.
- [7] V. P. Leonov and A. N. Shiryaev. "On a Method of Calculation of Semi-Invariants," *Theory of Probability and Its Applications*, Vol. 4, pp. 319-328, 1959.
- [8] C. M. Spooner. "Theory and Application of Higher-Order Cyclostationarity," Ph.D. Dissertation, Dept. of Electrical and Computer Engineering, University of California, Davis, CA, June 1992.